Conformal Covariance and Positivity of Energy in Charged Sectors

MIHÁLY WEINER
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, I-00133, Roma, ITALY
E-mail: weiner@mat.uniroma2.it

Abstract

It has been recently noted that the diffeomorphism covariance of a Chiral Conformal QFT in the vacuum sector automatically ensures Möbius covariance in all charged sectors. In this article it is shown that the diffeomorphism covariance and the positivity of the energy in the vacuum sector even ensure the positivity of the energy in the charged sectors.

The main observation of this paper is that the positivity of the energy — at least in case of a Chiral Conformal QFT — is a local concept: it is related to the fact that the energy density, when smeared with some local nonnegative test functions, remains bounded from below (with the bound depending on the test function).

The presented proof relies in an essential way on recently developed methods concerning the smearing of the stress-energy tensor on nonsmooth functions.

1 Introduction

The positivity of the energy is one of the most important selection criteria for a model to be "physical". In almost all treatments of Quantum Field

Theory, it appears as one of the fundamental axioms. In the vacuum sector it is usually formulated by requiring the positivity of the selfadjoint generator of every one-parameter group of future-like spacetime translations in the representation corresponding to the model.

As an axiom, one may say that it is "automatically" true, but in a concrete model it is something to be checked. In particular, to see what are the charged sectors with positive energy for a model given in the vacuum sector may be difficult (as calculating the charged sectors can already be a hard problem).

The present paper concerns chiral components of 2-dimensional Conformal QFTs in the setting of Algebraic Quantum Field Theory (see the book [19] of Haag). In this framework a Chiral Conformal QFT is commonly described by means of a Möbius covariant local net of von Neumann algebras on S^1. The net is said to be conformal (or diffeomorphism) covariant if the Möbius symmetry of the net extends to a full diffeomorphism symmetry (see the next section for precise definitions). Charged sectors are described as irreducible representations of the net; their general theory was developed by Doplicher, Haag and Roberts [9, 10]. They proved, among many other things, that if a covariant sector has a finite statistical dimension then it is automatically of positive energy. In [16] Guido and Longo showed that under some regularity condition the finiteness of the statistics even implies the covariance property of the sector.

In the particular case of chiral theories, there are many beautiful known relations concerning charged sectors; e.g. the formula [24, Theorem 33] of Kawahigashi, Longo e Müger, linking the statistical dimensions of the sectors to the so-called μ-index of the net. In relation with the positivity of energy we may say that the case of finite statistics is more or less completely understood [16, 17, 1]. In particular, taking in account the above mentioned formula, if a theory (in its vacuum sector) is split, conformal and has a finite μ-index — which means that it is *completely rational* cf. [24, 28] — then every sector of it is automatically of finite statistics and covariant under a positive energy representation of the Möbius group. Yet, although these conditions cover many of the interesting cases (for example all $\mathrm{SU}(N)_k$ models [31] and all models with central charge $c < 1$, see [23]), there are interesting (not pathological!) models in which it does not hold and, what is more important in this context, indeed possessing sectors with infinite statistical dimension (and yet with positivity of energy). This is clearly in contrast with the experience coming from massive QFTs (by a theorem of Buchholz and Fredenhagen [2], a massive sector with positive energy is always localizable

in a spacelike cone and has finite statistics).

The first example of a sector with infinite statistics was constructed by Fredenhagen [12]. Rehren gave arguments [30] that even the Virasoro model, which is in some sense the most natural model, should admit sectors with infinite statistical dimensions when its central charge $c \geq 1$ and that in fact in this case "most" of its sectors should be of infinite statistics. This was then actually proved [4] by Carpi first for the case $c = 1$ and then [5] for many other values of the central charge, leaving open the question only for some values of c between 1 and 2. Moreover Longo e Xu proved [28] that if $\mathcal{A}$ is a split conformal net with $\mu = \infty$ then $(\mathcal{A} \otimes \mathcal{A})^{\mathrm{flip}}$ has at least one sector with infinite statistical dimension, showing that the case of infinite statistics is indeed quite general.

Recently D'Antoni, Fredenhagen and Köster published a letter [7] with a proof that diffeomorphism covariance itself (in the vacuum sector) is already enough to ensure Möbius covariance in every (not necessary irreducible) representation: there always exists a unique (projective, strongly continuous) inner implementation of the Möbius symmetry. (In Prop. 3.3 we shall generalize this statement to the *n-Möbius* group, which is a natural realization of the n^{th} cover of the Möbius group in the group of all diffeomorphisms.) Thus the concept of the conformal energy, as the selfadjoint generator of rotations in a given charged sector, is at least well-defined. (Without the assumption of diffeomorphism covariance it is in general not true: there are Möbius covariant nets — see the examples in [18, 25] — possessing charged sectors in which the Möbius symmetry is not even implementable.) What remained an open question until now, whether this energy is automatically positive or not. The present article shall settle this question by providing a proof for the positivity (Theorem 3.8).

The idea behind the proof is simple. The total conformal energy L_0 is the integral of the energy-density; i.e. the stress-energy tensor T evaluated on the constant 1 function. So if we take a finite partition of the unity $\{f_n\}_{n=1}^{N}$ on the circle, we may write $T(1)$ as the sum $\sum T(f_n)$ where each element is *local*. Thus each term in itself (although not bounded) can be considered in a given charged sector. Moreover, it has been recently proved by Fewster and Holland [11] that the stress-energy tensor evaluated on a nonnegative function is bounded from below. These operators then, being local elements, remain bounded from below also in the charged sector. So their sum in the charged sector, which we may expect to be the generator of rotations in that sector, should still be bounded from below.

There are several problems with this idea. For example, as the supports of the functions $\{f_n\}$ must unavoidably "overlap", the corresponding operators will in general not commute. To deal with sums of non-commuting unbounded operators is not easy. In particular, while in the vacuum representation — due to the well known energy bounds — we have the natural common domain of the finite energy vectors, in a charged sector (unless we assume positivity of energy, which is exactly what we want to prove) we have no such domain.

To overcome the difficulties we shall modify this idea in two points. First of all, instead of $L_0 = T(1)$, that is, the generator of the rotations, we may work with the generator of the translations — the positivity of any of them implies the positivity of the other one. In fact we shall go one step further by replacing the generator of translations with the generator of 2-*translations*. (This is why, as it has been already mentioned, we shall consider the n-Möbius group; particularly in the case $n = 2$.) This has the advantage that the function representing the corresponding vector field can be written as $f_1 + f_2$, where the two local nonnegative functions f_1, f_2 *do not* "overlap". These functions, at the endpoint of their support are not smooth (such decomposition is not possible with smooth functions); they are only once differentiable. However, as it was recently proved [6] by the present author and Carpi, the stress-energy tensor can be evaluated even on nonsmooth functions, given that they are "sufficiently regular", which is exactly the case of f_1 and f_2 (see Lemma 2.2 and the argument before Prop. 3.2). As they are nonnegative but not smooth, to conclude that $T(f_1)$ and $T(f_2)$ are bounded from below we cannot use the result stated in [11]. However, it turns out to be (Prop. 3.2) a rather direct and simple consequence of the construction, thus it will be deduced independently from the mentioned result, of which we shall make no explicit use. In fact the author considered this construction as an argument indicating that if $f \geq 0$ then $T(f)$ is bounded from below (which by now is of course proven, as it was already mentioned, in [11]); see more on this in this paper at the remark after Prop 3.2 and in the mentioned article of Fewster and Holland at the footnote in the proof of [11, Theorem 4.1].

Before we shall proceed to the proof, in the next section we briefly recall some definitions and basic facts regarding local nets of von Neumann algebras on the circle.

2 Preliminaries

2.1 Möbius covariant nets and their representations

Let $\mathcal{I}$ be the set of open, nonempty and nondense arcs (intervals) of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. A **Möbius covariant net on** S^1 is a map $\mathcal{A}$ which assigns to every $I \in \mathcal{I}$ a von Neumann algebra $\mathcal{A}(I)$ acting on a fixed complex, Hilbert space $\mathcal{H}_{\mathcal{A}}$ ("the vacuum Hilbert space of the theory"), together with a given strongly continuous representation U of Möb $\simeq$ PSL$(2,\mathbb{R})$, the group of Möbius transformations[1] of the unit circle S^1 satisfying for all $I_1, I_2, I \in \mathcal{I}$ and $\varphi \in$ Möb the following properties.

(i) *Isotony.* $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

(ii) *Locality.* $I_1 \cap I_2 = \emptyset \Rightarrow [\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$.

(iii) *Covariance.* $U(\varphi)\mathcal{A}(I)U(\varphi)^{-1} = \mathcal{A}(\varphi(I))$.

(iv) *Positivity of the energy.* The representation U is of positive energy type: the conformal Hamiltonian L_0, defined by $U(R_\alpha) = e^{i\alpha L_0}$ where $R_\alpha \in$ Möb is the anticlockwise rotation by an angle of α, is positive.

(v) *Existence and uniqueness of the vacuum.* Up to phase there exists a unique unit vector $\Omega \in \mathcal{H}_{\mathcal{A}}$ called the "vacuum vector" which is invariant under the action of U.

(vi) *Cyclicity of the vacuum.* Ω is cyclic for the von Neumann algebra $\mathcal{A}(S^1) \equiv \{\mathcal{A}(I) : I \in \mathcal{I}\}''$.

There are many known consequences of the above listed axioms. We shall recall some of the most important ones referring to [14, 17] and [13] for proves. 1. *Reeh-Schlieder property:* Ω is a cyclic and separating vector of the algebra $\mathcal{A}(I)$ for every $I \in \mathcal{I}$. 2. *Bisognano-Wichmann property:* $U(\delta^I_{2\pi t}) = \Delta_I^{it}$ where Δ_I is the modular operator associated to $\mathcal{A}(I)$ and Ω, and δ^I is the one-parameter group of Möbius transformations preserving the interval I (the dilations associated to I) with parametrization fixed in the beginning of the next section. 3. *Haag-duality:* $\mathcal{A}(I)' = \mathcal{A}(I^c)$ for every $I \in \mathcal{I}$, where I^c denotes the interior of the complement set of I in S^1.

[1] diffeomorphisms of S^1 of the form $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$ with $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$.

4. *irreducibility*: $\mathcal{A}(S^1) = \mathrm{B}(\mathcal{H}_\mathcal{A})$, where $\mathrm{B}(\mathcal{H}_\mathcal{A})$ denotes the algebra of all bounded linear operators on $\mathcal{H}_\mathcal{A}$. 5. *factoriality*: for an $I \in \mathcal{I}$ the algebra $\mathcal{A}(I)$ is either just the trivial algebra $\mathbb{C}\mathbb{1}$ (in which case $\dim(\mathcal{H}_\mathcal{A}) = 1$ and the whole net is trivial) or it is a type III_1 factor for every $I \in \mathcal{I}$. 6. *additivity*: if $\mathcal{S} \subset \mathcal{I}$ is a covering of the interval I then $\mathcal{A}(I) \subset \{\mathcal{A}(J) : J \in \mathcal{S}\}''$. Note that by the *Bisognano-Wichmann property*, since Möb is generated by the dilations (associated to different intervals), the representation U is completely determined by the local algebras and the vacuum vector via modular structure.

A **locally normal representation** π (or for shortness, just simply representation) of a Möbius covariant local net $(\mathcal{A}, U)$ consits of a Hilbert space $\mathcal{H}_\pi$ and a normal representation π_I of the von Neumann algebra $\mathcal{A}(I)$ on $\mathcal{H}_\pi$ for each $I \subset \mathcal{I}$ such that the collection of representations $\{\pi_I : I \in \mathcal{I}\}$ is consistent with the *isotony*: $I \subset K \Rightarrow \pi_K|_{\mathcal{A}(I)} = \pi_I$. It follows easily from the axioms and the known properties of local nets listed above that if $I \cap K = \emptyset$ then $[\pi_I(\mathcal{A}(I)), \pi_K(\mathcal{A}(K))] = 0$, if $\mathcal{S} \subset \mathcal{I}$ is covering of $K \in \mathcal{I}$ then $\{\pi_I(\mathcal{A}(I)) : I \in \mathcal{I}\}'' \supset \pi_K(\mathcal{A}(K))$, if $\mathcal{S} \subset \mathcal{I}$ is a covering of S^1 then $\{\pi_I(\mathcal{A}(I)) : I \in \mathcal{S}\}'' = \{\pi_I(\mathcal{A}(I)) : I \in \mathcal{I}\}'' \equiv \pi(\mathcal{A})$ and finally, that for each $I \in \mathcal{I}$ the representation π_I is faithful. The representation π is called *irreducible*, if $\pi(\mathcal{A})' = \mathbb{C}\mathbb{1}$.

2.2 Diffeomorphism covariance

Let $\mathrm{Diff}^+(S^1)$ be the group of orientation preserving (smooth) diffeomorphisms of the circle. It is an infinite dimensional Lie group whose Lie algebra is identified with the real topological vector space $\mathrm{Vect}(S^1)$ of smooth real vector fields on S^1 with the usual C^∞ topology [29, Sect. 6] with the negative[2] of the usual bracket of vector fields. We shall think of a the vector field symbolically written as $f(e^{i\vartheta})\frac{d}{d\vartheta} \in \mathrm{Vect}(S^1)$ as the corresponding real function f. We shall use the notation f' (calling it simply the derivative) for the function on the circle obtained by derivating with respect to the angle: $f'(e^{i\theta}) = \frac{d}{d\alpha} f(e^{i\alpha})|_{\alpha=\theta}$.

A strongly continuous projective unitary representation V of $\mathrm{Diff}^+(S^1)$ on a Hilbert space $\mathcal{H}$ is a strongly continuous $\mathrm{Diff}^+(S^1) \to \mathcal{U}(\mathcal{H})/\mathbb{T}$ homomorphism. The restriction of V to $\text{Möb} \subset \mathrm{Diff}^+(S^1)$ always lifts to a unique

[2]The negative sign is "compulsory" if we want the "abstract" exponential — defined for Lie algebras of Lie groups — to be the same as the exponential of vector fields, i.e. the diffeomorphism which is the generated flow at time equal 1.

strongly continuous unitary representation of the universal covering group $\widetilde{\text{Möb}}$ of Möb. V is said to be of positive energy type, if its conformal Hamiltonian L_0, defined by the above representation of $\widetilde{\text{Möb}}$ (similarly as in case of a representation of the group Möb) has nonnegative spectrum.

Sometimes for a $\gamma \in \text{Diff}^+(S^1)$ we shall think of $V(\gamma)$ as a unitary operator. Although there are more than one way to fix phases, note that expressions like $\text{Ad}(V(\gamma))$ or $V(\gamma) \in \mathcal{M}$ for a von Neumann algebra $\mathcal{M} \subset \text{B}(\mathcal{H})$ are unambiguous. Note also that the selfadjoint generator of a one-parameter group of strongly continuous *projective* unitaries $t \mapsto Z(t)$ is well defined up to a real additive constant: there exists a selfadjoint operator A such that $\text{Ad}(Z(t)) = \text{Ad}(e^{iAt})$ for all $t \in \mathbb{R}$, and if A' is another selfadjoint with the same property then $A' = A + r\mathbb{1}$ for some $r \in \mathbb{R}$.

We shall say that V is an extension of the unitary representation U of Möb if we can arrange the phases in such a way that $V(\varphi) = U(\varphi)$, or without mentioning phases: $\text{Ad}(V(\varphi)) = \text{Ad}(U(\varphi))$, for all $\varphi \in$ Möb. Note that such an extension of a positive energy representation of Möb is of positive energy.

Definition 2.1. *A Möbius covariant net $(\mathcal{A}, U)$ is said to be* **conformal (or diffeomorphism) covariant** *if there is a strongly continuous projective unitary representation of* $\text{Diff}^+(S^1)$ *on* $\mathcal{H}_{\mathcal{A}}$ *which extends* U *(and by an abuse of notation we shall denote this extension, too, by* U*), and for all* $\gamma \in \text{Diff}^+(S^1)$ *and* $I \in \mathcal{I}$ *satisfies*

- $U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma(J))$,
- $\gamma|_I = \text{id}_I \Rightarrow \text{Ad}(\text{U}(\gamma))|_{\mathcal{A}(\text{I})} = \text{id}_{\mathcal{A}(\text{I})}$.

Note that as a consequence of *Haag duality*, if a diffeomorphism is localized in the interval I — i.e. it acts trivially (identically) elsewhere — then, by the second listed property the corresponding unitary is also localized in I in the sense that it belongs to $\mathcal{A}(I)$. Thus by setting

$$\mathcal{A}_U(I) \equiv \{U(\gamma) : \gamma \in \text{Diff}^+(S^1), \gamma|_{I^c} = \text{id}_{I^c}\}'' \quad (I \in \mathcal{I}) \tag{1}$$

we obtain a *conformal subnet*: for all $\gamma \in \text{Diff}^+(S^1)$ and $I \in \mathcal{I}$ we have that $\mathcal{A}_U(I) \subset \mathcal{A}(I)$ and $U(\gamma)\mathcal{A}_U(I)U(\gamma)^* = \mathcal{A}_U(\gamma(I))$. The restriction of the subnet $\mathcal{A}_U$ onto the Hilbert space $\mathcal{H}_{\mathcal{A}_U} \equiv \overline{(\bigvee_{I\in\mathcal{I}} \mathcal{A}_U(I))\Omega}$ is again a conformal net, which — unless $\mathcal{A}$ is trivial — by [5, Theorem A.1] is isomorphic to a so-called *Virasoro net*. For a representation π of $\mathcal{A}$ we set $\pi(\mathcal{A}_U) \equiv \{\pi_I(\mathcal{A}_U(I)) : I \in \mathcal{I}\}''$.

The smooth function $f : S^1 \to \mathbb{R}$, as a vector field on S^1, gives rise to the one-parameter group of diffeomorphisms $t \mapsto \mathrm{Exp}(tf)$. Hence, up to an additive real constant the selfadjoint generator $T(f)$ of $t \mapsto U(\mathrm{Exp(tf)})$ is well defined. For any real smooth function f on the circle $T(f)$ is essentially selfadjoint on the dense set of *finite-energy vectors*, i.e. on the algebraic span of the eigenvectors of L_0. By the condition $< \Omega, T(\cdot)\Omega >= 0$ fixing the additive constant in its definition, T is called the **stress-energy tensor** associated to U. It is an operator valued linear functional in the sense that on the set of finite energy vectors $T(f + \lambda g) = T(f) + \lambda T(g)$ for all f, g real smooth functions on the circle and $\lambda \in \mathbb{R}$. Note that by the second listed condition in Def. 2.1 if $\mathrm{Supp}(f) \subset I$ for a certain $I \in \mathfrak{I}$ then $T(f)$ is affiliated to $\mathcal{A}(I)$.

For a more detailed introduction on the stress-energy tensor see for example [6, 5]. The proof of the statements made in defining T relies on the so-called Virasoro operators, which can always be introduced (see the remarks in the beginning of [6, Sect. 4] and before [5, Theorem A.1], all using the results [27] of Loke), and on the existence of some "energy bounds" (see [15, 3]).

In this paper we shall often use nonsmooth functions. For a function $f \in C(S^1, \mathbb{R})$ with Fourier coefficients $\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha} f(e^{i\alpha})\, d\alpha$ $(n \in \mathbb{Z})$ we set

$$\|f\|_{\frac{3}{2}} = \sum_{n\in\mathbb{Z}} |\hat{f}_n|(1 + |n|^{\frac{3}{2}}) \in \mathbb{R}_0^+ \cup \{+\infty\}. \tag{2}$$

In [6, Sect. 4] the present author with Carpi proved that T can be continuously extended to functions with finite $\|\cdot\|_{\frac{3}{2}}$ norm as

- if f, f_n $(n \in \mathbb{N})$ are real smooth functions on the circle and $f_n \to f$ in the $\|\cdot\|_{\frac{3}{2}}$ sense then $T(f_n)$ converges to $T(f)$ in the strong resolvent sense,

- if f_n $(n \in \mathbb{N})$ is a Cauchy sequence of real smooth functions with respect to the $\|\cdot\|_{\frac{3}{2}}$ norm then $T(f_n)$ converges to a selfadjoint operator in the strong resolvent sense, which is essentially selfadjoint on the finite energy vectors,

- the real smooth functions form a dense set among the real continuous functions with finite $\|\cdot\|_{\frac{3}{2}}$ norm.

Thus one can consider $T(f)$ even when f is not smooth but its $\|\cdot\|_{\frac{3}{2}}$ norm is finite. The following lemma, which was essentially demonstrated in the proof of [6, Lemma 5.3] but was not stated there can be useful in some cases to establish the finiteness of this norm.

Lemma 2.2. *Let f be a (once) differentiable function on the circle. Suppose that there exists a finite set of intervals $I_k \in \mathfrak{I}$ and smooth functions g_k on the circle $(k = 1, .., N)$ such that $\overline{\cup_{k=1}^N I_k} = S^1$ and $f|_{I_k} = g_k|_{I_k}$. Then $\|f\|_{\frac{3}{2}} < \infty$.*

Proof. The conditions mean that f'', which is everywhere defined apart from a finite set of points, has Fourier coefficients $(\hat{f''})_n = -n^2 \hat{f}_n$ and is of bounded variation. Therefore $|n^2 \hat{f}_n| \leq |\frac{\mathrm{Var}(f'')}{n}|$ (see [22, Sect. I.4]), from which the claim follows easily. □

In relation with the net $(\mathcal{A}, U)$ the extension to nonsmooth functions is still *covariant* and *local* in the sense of the following statement (which again was essentially proved in [6], but was not explicitly stated there).

Proposition 2.3. *Let $\gamma \in \mathrm{Diff}^+(S^1)$ and f be a real continuous function on the circle with both $\|f\|_{\frac{3}{2}} < +\infty$ and $\|\gamma_* f\|_{\frac{3}{2}} < +\infty$ where γ_* stand for the action of γ on vector fields. Then up to phase factors*

$$U(\gamma)\, e^{iT(f)}\, U(\gamma)^* = e^{iT(\gamma_* f)}.$$

Moreover, if Supp$(f) \subset \overline{I}$ where $I \in \mathfrak{I}$, then $T(f)$ is affiliated to $\mathcal{A}(I)$.

Proof. For the second part of the statement, by the continuity [20] of the net we may assume that $\mathrm{Supp}(f)$ is already contained in I (and not only in its closure). Then according to [6, Lemma 4.6], there exists a sequence of smooth functions f_n $(n \in \mathbb{N})$ converging to f in the $\|\cdot\|_{\frac{3}{2}}$ norm whose support is contained in I. Then, by [6, Prop. 4.5] $T(f_n)$ converges to $T(f)$ in the strong resolvent sense, and thus $T(f)$ is affiliated to $\mathcal{A}(I)$ as $T(f_n)$ is affiliated to $\mathcal{A}(I)$ for each $n \in \mathbb{N}$.

The first part of the statement is again obviously true if f is smooth, as then $e^{iT(f)} = U(\mathrm{Exp}(f))$ and $e^{iT(\gamma_* f)} =$

$$U(\mathrm{Exp}(\gamma_* f) = U(\gamma \circ \mathrm{Exp}(f) \circ \gamma^{-1}) = U(\gamma)\, U(\mathrm{Exp}(f))\, U(\gamma)^*. \qquad (3)$$

Then similarly to the first part, by approximating f with smooth functions and taking limits one can easily finish the proof. □

3 Proof of the Positivity

Apart from the subgroup Möb $\subset \mathrm{Diff}^+(S^1)$, for our argument we shall need to use some other important subgroups. For each positive integer n the group $\text{Möb}^{(n)}$ is defined to be the subgroup of $\mathrm{Diff}^+(S^1)$ containing all elements $\gamma \in \mathrm{Diff}^+(S^1)$ for which there exists a Möbius transformation $\phi \in$ Möb satisfying

$$\gamma(z)^n = \phi(z^n) \ \ (\forall z \in S^1). \tag{4}$$

Thus the group $\text{Möb}^{(n)}$ gives a natural n-covering of Möb. This group has been already considered and successfully used for the analyses of conformal nets, see e.g. [28].

In Möb, beside the rotations one often considers the translations $a \mapsto \tau_a$ and the dilations $s \mapsto \delta_s$, that are the one-parameter groups generated by the vector fields $t(z) = \frac{1}{2} - \frac{1}{4}(z + z^{-1})$ and $d(z) = \frac{i}{2}(z + z^{-1})$, respectively. For an $I \in \mathfrak{I}$ one may choose a transformation $\phi \in$ Möb such that $\phi(S^1_+) = I$ where $S^1_\pm = \{z \in S^1 : \pm\mathrm{Im}(z) > 0\}$. The one-parameter group $s \mapsto \phi \circ \delta_s \circ \phi^{-1}$ is independent of ϕ (see e.g. [18]) and is called the dilations associated to the interval I. When no interval is specified, δ always stands for the one associated to S^1_+.

By direct calculation $[d, t] = t$ (remember that the bracket is the negative of the usual bracket of vector fields) and thus at the group level we find

$$\delta_s \tau_a \delta_{-s} = \tau_{e^s a} \tag{5}$$

i.e. the dilations "scale" the translations.

In $\text{Möb}^{(n)}$, just like in Möb, one introduces the one-parameter subgroup of translations $a \mapsto \tau^{(n)}_a$, which is defined by the usual procedure of lifting: it is the unique continuous one-parameter subgroup satisfying $\tau^{(n)}_a(z)^n = \tau_a(z^n)$. Alternatively, one may define it directly with its generating vector field $t^{(n)}(z) = \frac{1}{2n} - \frac{1}{4n}(z^n + z^{-n})$. Similarly one introduces the notion of rotations $\alpha \mapsto R^{(n)}_\alpha$ and of dilations $s \mapsto \delta^{(n)}_s$. Of course the "n-rotations", apart from a rescaling of the parameter, will simply coincide with the "true" rotations:

$$R^{(n)}_\alpha = R_{\alpha/n}. \tag{6}$$

Let us now consider a strongly continuous projective unitary representation $V^{(n)}$ of $\text{Möb}^{(n)}$. The group $\text{Möb}^{(n)}$ is connected and its Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ which is in particular semisimple (in fact even simple, but for what follows semisimplicity is enough). Therefore, as it is well

known, the representation $V^{(n)}$ has a unique strongly continuous lift $\tilde{V}$ to the universal cover of $\text{Möb}^{(n)}$ which is a *true* representation. As $\text{Möb}^{(n)}$ covers Möb in a natural way, its universal cover is canonically identified with $\widetilde{\text{Möb}}$ which is isomorphic to $\widetilde{\text{SL}(2,\mathbb{R})}$.

The following lemma, although contains some well known facts, is hereby included for the convenience of the reader. The presented proof is an adopted (and slightly modified) version of the proof of [26, Prop. 1].

Lemma 3.1. *Let $\tilde{V}$ be a strongly continuous unitary representation of $\widetilde{\text{Möb}}$ with H and P being the selfadjoint generator of rotations and translations in $\tilde{V}$, respectively. Then the following four conditions are equivalent:*

1. *H is bounded from below,*
2. *P is bounded from below,*
3. *$H \geq 0$,*
4. *$P \geq 0$.*

Proof. Let $\tilde{R}$ be the lift of R and set $P_\pi = \tilde{V}(\tilde{R}_\pi)P\tilde{V}(\tilde{R}_\pi)^*$; it is then the selfadjoint generator associated to the one-parameter group generated by the vector field t_π which we get by rotating t by π radian i.e. $t_\pi(z) = \frac{1}{2}+\frac{1}{4}(z+z^{-1})$. As P_π is unitary conjugate to P their spectra coincide. Moreover, as $t+t_\pi = 1$ on the Gårding-domain we have that $P + P_\pi = H$ which immediately proves that if P is bounded from below or positive then so is H.

As for the rest of the statement, apart from the trivial indications there remain only to show that if H is bounded from below then P is positive. Consider the lifted dilations $s \mapsto \tilde{\delta}_s$. By equation (5) one has that $\tilde{V}(\tilde{\delta}_s)P\tilde{V}(\tilde{\delta}_s)^* = e^s P$. Moreover, by direct calculation $[d, t_\pi] = -t_\pi$ so similarly to the case of translations the dilations also "scale" t_π, but in the converse direction. Thus $\tilde{V}(\tilde{\delta}_s)P_\pi\tilde{V}(\tilde{\delta}_s)^* = e^{-s}P_\pi$. So if $H \geq r\mathbb{1}$ for some r real (but not necessarily nonnegative) number then for any vector ξ in the Gårding-domain, setting $\eta = \tilde{V}(\tilde{\delta}_s)^*\xi$ we have that

$$r\|\xi\|^2 = r\|\eta\|^2 \quad \leq \quad <\eta, H\eta> = e^s <\xi, P\xi> + e^{-s} <\xi, P_\pi\xi> \qquad (7)$$

from which, letting $s \to \infty$ we find that $P \geq 0$. □

If any of the conditions of the above lemma is satisfied, $\tilde{V}$ is called a positive energy representation. A projective representation $V^{(n)}$ of $\text{Möb}^{(n)}$ is said to be of positive energy if its lift to $\widetilde{\text{Möb}}$ is of positive energy.

Let us now consider a conformal local net of on the circle $(\mathcal{A}, U)$. By equation (6), $U^{(n)}$, the restriction of the positive energy representation of

U of $\mathrm{Diff}^+(S^1)$ with stress-energy tensor T, is a positive energy projective representation of $\mathrm{Möb}^{(n)}$. In particular, as $U^{(2)}$ is of positive energy, the selfadjoint operator $T(t^{(2)})$ must be bounded from below, since it generates the translations for the representation $U^{(2)}$. (Note that $T(t^{(2)})$ is bounded from below but not necessary positive: it is not *the* generator — it still generates the same projective one-parameter group of unitaries if you add a real constant to it.) The function $t^{(2)}(z) = \frac{1}{4} - \frac{1}{8}(z^2 + z^{-2})$ is a nonnegative function with two points of zero: $t^{(2)}(\pm 1) = 0$. By direct calculation of the first derivative: $(t^{(2)})'(\pm 1) = 0$, hence the decomposition

$$t^{(2)} = t^{(2)}_+ + t^{(2)}_- \tag{8}$$

with the functions $t^{(2)}_\pm$ defined by the condition $\mathrm{Supp}(t^{(2)}_\pm) = (S^1_\mp)^c$ is a decomposition of $t^{(2)}$ into the sum of two (once) differentiable nonnegative functions that satisfy the conditions of Lemma 2.2. Therefore, as it was explained in the Preliminaries, we can consider the selfadjoint operators $T(t^{(2)}_\pm)$.

Proposition 3.2. *Let $(\mathcal{A}, U)$ be a conformal net of local algebras on the circle with stress-energy tensor T. Then $T(t^{(2)}_+)$ is affiliated to $\mathcal{A}(S^1_+)$ and $T(t^{(2)}_-)$ is affiliated to $\mathcal{A}(S^1_-)$ and so in particular they strongly commute. Moreover, the operators $T(t^{(2)}_\pm)$ are bounded from below.*

Proof. $\mathrm{Supp}(t^{(2)}_\pm) \subset \overline{S^1_\pm}$ and so by Prop. 2.3 $T(t^{(2)}_\pm)$ is affiliated to $\mathcal{A}(S^1_\pm)$. So if $P_{[a,b]}$ is a nonzero spectral projection of $T(t^{(2)}_+)$ and $Q_{[c,d]}$ is a nonzero spectral projection of $T(t^{(2)}_-)$, then $P_{[a,b]} \in \mathcal{A}(S^1_+)$, $Q_{[a,b]} \in \mathcal{A}(S^1_-)$ and by the algebraic independence of two commuting factors (see for example [21, Theorem 5.5.4]) $R = P_{[a,b]}Q_{[c,d]} \neq 0$. Of course the range of R is invariant for (and included in the domain of) $T(t^{(2)}_+) + T(t^{(2)}_-)$ and the restriction of that operator for this closed subspace is clearly bigger than $a + c$ and smaller than $b + d$. Thus

$$\mathrm{Sp}\left(T(t^{(2)}_+) + T(t^{(2)}_-)\right) \supset \mathrm{Sp}(T(t^{(2)}_+)) + \mathrm{Sp}(T(t^{(2)}_-)). \tag{9}$$

To conclude we only need to observe that by equation (8) on the common core of the finite energy vectors $T(t^{(2)}_+) + T(t^{(2)}_-) = T(t^{(2)})$, and as it was said, the latter selfadjoint operator is bounded from below. $\square$

Remark. The author considered this construction to indicate that if $f \geq 0$ then $T(f)$ is bounded from below, which — as it was already mentioned —

by now is a proven fact (cf.[11]). The point is the following. If f is *strictly* positive then, as a vector field on S^1, it is conjugate to the constant vector field r for some $r > 0$. Thus, using the transformation rule of T under diffeomorphisms, $T(f)$ is conjugate to $T(r)$ plus a constant, and so it is bounded from below by the positivity of $T(1) = L_0$. The real question is whether the statement remains true even when f is nonnegative, but not strictly positive because for example it is *local* (there is an entire interval on which it is zero). One can of course consider a nonnegative function as a limit of positive functions, but then one needs to control that the lowest point of the spectrum does not go to $-\infty$ while taking this limit (which — in a slightly different manner — has been successfully carried out in the mentioned article). However, even without considering limits, by the above proposition we find nontrivial examples of local nonnegative functions g such that $T(g)$ can easily be checked to be bounded from below. (Take for example $g = t^{(2)}_{\pm}$ but of course we may consider conjugates, sums and multiples by positive constants to generate even more examples.)

Let us now investigate what we can say about a representation π of the conformal net $(\mathcal{A}, U)$. In [7] it was proved that the Möbius symmetry is continuously implementible in any (locally normal) representation π by a unique inner projective way. By their construction the implementing operators are elements of $\pi(\mathcal{A}_U)$. Looking at the article, we see that the only structural properties of the Möbius subgroup of $\mathrm{Diff}^+(S^1)$ that the proof uses are the following.

- There exist three continuous one-parameter groups Γ_1, Γ_2 and Γ_3 in Möb, so that every element $\gamma \in$ Möb can be uniquely written as a product $\gamma = \Gamma_1(s_1)\Gamma_2(s_2)\Gamma_3(s_3)$ where the parameters (s_1, s_2, s_3) depend continuously on γ. (In the article Γ_1 is the translational, Γ_2 is the dilational and Γ_3 is the rotational subgroup; which is the so-called Iwasawa decomposition, see [14].)

- The Lie algebra of Möb is simple.

These two properties hold not only for the subgroup Möb, but also for $\text{Möb}^{(n)}$ where n is any positive integer: for all n the Lie algebra of $\text{Möb}^{(n)}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, and with the rotations, dilations and translations replaced by n-rotations, n-dilations and n-translations we still have the required decomposition. Let us collect into a proposition what we have thus concluded.

Proposition 3.3. *Let π be a locally normal representation of the conformal local net of von Neumann algebras on the circle $(\mathcal{A}, U)$. Then for all positive integer n there exists a unique strongly continuos projective representation $U_\pi^{(n)}$ of $\text{Möb}^{(n)}$ such that $U_\pi^{(n)}(\text{Möb}^{(n)}) \subset \pi(\mathcal{A})$ and for all $\gamma \in \text{Möb}^{(n)}$ and $I \in \mathfrak{I}$*

$$\text{Ad}(U^{(n)}(\gamma)) \circ \pi_I = \pi_{\gamma(I)} \circ \text{Ad}(U^{(n)}(\gamma)).$$

Moreover, this unique representation satisfies $U_\pi^{(n)}(\text{Möb}^{(n)}) \subset \pi(\mathcal{A}_U)$.

We shall now return to the particular case $n = 2$. On one hand, the action of the 2-translation $\tau_a^{(2)}$ in the representation π can be implemented by $U_\pi^{(2)}(\tau_a^{(2)})$. On the other hand, as

$$U(\tau_a^{(2)}) = e^{iaT(t^{(2)})} = e^{iaT(t_+^{(2)})} e^{iaT(t_-^{(2)})} \tag{10}$$

we may try to implement the same action by $\pi_{S_+^1}(W_+(a))\pi_{S_-^1}(W_-(a))$, where

$$W_\pm(a) = e^{iaT(t_\pm^{(2)})} \in \mathcal{A}_U(S_\pm^1). \tag{11}$$

Proposition 3.4. *The unitary operator in $\pi(\mathcal{A}_U)$*

$$W_\pi(a) := \pi_{S_+^1}(W_+(a))\, \pi_{S_-^1}(W_-(a)) = \pi_{S_-^1}(W_-(a))\, \pi_{S_+^1}(W_+(a))$$

up to phase coincides with $U_\pi^{(2)}(\tau_a^{(2)})$.

Proof. It is more or less trivial that the adjoint action of the two unitaries coincide on both $\pi_{S_+^1}(\mathcal{A}(S_+^1))$ and $\pi_{S_-^1}(\mathcal{A}(S_-^1))$. There remain two problems to overcome:

- the algebra generated by these two algebras do not necessarily contain $\pi(\mathcal{A}_U)$, so it is not clear why the adjoint action of these two unitaries should coincide on the mentioned algebra,
- but even if we knew that the actions coincide, the two unitaries, although both belonging to $\pi(\mathcal{A}_U)$, for what we know could still "differ" in an inner element.

As for the first problem, consider an open interval $I \subset S^1$ such that it contains the point -1 and has 1 in the interior of its complement. Note that due to the conditions imposed on I, the sets $K_\pm := I \cup S_\pm^1$ are still elements of $\mathfrak{I}$.

Lemma 3.5. *If $a \geq 0$ then $W_+(a)\mathcal{A}(I)W_+(a)^* \subset \mathcal{A}(I)$.*

Proof of the Lemma. Let us take a sequence of nonnegative smooth functions ϕ_n $(n = 1, 2, ..)$ on the real line, such that the support of ϕ_n is contained in the interval $(-1/n, 1/n)$, and its integral is 1. Then, exactly as in [6, Prop. 4.5, Lemma 4.6], we have that $T(\rho_n)$, with ρ_n being the convolution

$$\rho_n(e^{i\theta}) \equiv (t_+^{(2)} * \phi_n)(e^{i\theta}) \equiv \int t_+^{(2)}(e^{i(\theta+\alpha)})\phi_n(\alpha)\, d\alpha, \tag{12}$$

converges to $T(t_+^{(2)})$ in the strong resolvent sense.

The flow of a vector field given by a nonnegative function on the circle, moves all points forward (i.e. anticlockwise). Moreover, the flow cannot move points from the support of the vector field to outside, and leaves invariant all points outside.

The function ρ_n — being the convolution of two nonnegative function — is nonnegative, and its support is S^1_+ "plus $1/n$ radius in both direction". Taking in consideration what was said before it is easy to see that for n large enough $\mathrm{Exp}(a\rho_n)(I) \subset I$ and consequently

$$\mathrm{Ad}\left(e^{iaT(\rho_n)}\right)(\mathcal{A}(I)) \subset \mathcal{A}(I). \tag{13}$$

Then by the convergence in the strong resolvent sense we obtain what we have claimed. $\square$

It follows that if $A \in \mathcal{A}(I)$ and $a \geq 0$ then

$$\begin{aligned} &\pi_{S^1_+}(W_+(a))\, \pi_I(A)\, \pi_{S^1_+}(W_+(a))^* = \\ &\pi_{K_+}(W_+(a)\, A\, W_+(a)^*) = \pi_I(W_+(a)\, A\, W_+(a)^*) \end{aligned} \tag{14}$$

and thus $\mathrm{Ad}\,(\mathrm{W}_\pi(\mathrm{a}))\,(\pi_\mathrm{I}(\mathrm{A})) = \mathrm{Ad}\left(\pi_{\mathrm{S}^1_-}(\mathrm{W}_-(\mathrm{a}))\,\pi_{\mathrm{S}^1_+}(\mathrm{W}_+(\mathrm{a}))\right)(\pi_\mathrm{I}(\mathrm{A})) =$

$$\begin{aligned} &= \mathrm{Ad}\left(\pi_{\mathrm{S}^1_-}(\mathrm{W}_-(\mathrm{a}))\right)(\pi_\mathrm{I}(\mathrm{W}_+(\mathrm{a})\,\mathrm{A}\,\mathrm{W}_+(\mathrm{a})^*)) \\ &= \pi_{K_-}(W_-\,(W_+(a)\, A\, W_+(a)^*)\, W_-(a)^*) \\ &= \pi_{K_-} U(\tau_a^{(2)})\, A\, U(\tau_a^{(2)})^*) = \mathrm{Ad}\left(\mathrm{U}_\pi^{(2)}(\tau_\mathrm{a}^{(2)})\right)(\pi_\mathrm{I}(\mathrm{A})) \end{aligned} \tag{15}$$

where in the last equality we have used the fact that for $a \geq 0$ the image of I under the diffeomorphism $\tau_a^{(2)}$ is contained in K_-.

We have thus seen that for $a \geq 0$ the adjoint action of $W_\pi(a)$ and of $U_\pi^{(2)}(\tau_a^{(2)})$ coincide on $\pi_I(A(I))$. Actually, looking at our argument we can realize that everything remains true if instead of I we begin with an open interval L that contains the point 1 and has -1 in the interior of its complement and we exchange the "+" and "-" subindices. So in fact we have proved that for $a \geq 0$ these adjoint actions coincide on both $\pi_I(A(I))$ and $\pi_L(A(L))$ and therefore on the whole algebra $\pi(\mathcal{A})$, since we may assume that the union of I and L is the whole circle. (The choice of the intervals, apart from the conditions listed, was arbitrary.) Of course the equality of the actions, as they are obviously one-parameter automorphism groups of $\pi(\mathcal{A})$, is true also in case the parameter a is negative. We can now also confirm that the unitary

$$Z_\pi(a) \equiv W_\pi(a)^* \, U_\pi^{(2)}(\tau_a^{(2)}) \tag{16}$$

lies in $\mathcal{Z}(\pi(\mathcal{A})) \cap \pi(\mathcal{A}_U) \subset \mathcal{Z}(\pi(\mathcal{A}_U))$ where "$\mathcal{Z}$" stands for the word "center". Thus $a \mapsto Z_\pi(a)$ is a strongly continuous (projective) one-parameter group. (It is easy to see that as Z_π commutes with both W_π and $U_\pi^{(2)}$ it is actually a one-parameter group.)

We shall now deal with the second mentioned problem. The 2-dilations $s \mapsto \delta_s^{(2)}$ scale the 2-translations and preserve the intervals $S^1_\pm$. Thus they also scale the functions $t_\pm^{(2)}$ and so we get some relations — both in the vacuum and in the representation π — regarding the unitaries implementing the dilations and translations and the unitaries that were denoted by W with different subindices (see Prop. 2.3). More concretely, with everything meant in the projective sense, in the vacuum Hilbert space we have that the adjoint action of $U^{(2)}(\delta_s^{(2)})$ scales the parameter a into $e^s a$ in $U^{(2)}(\tau_a^{(2)})$ and in $W_\pm(a)$ while in $\mathcal{H}_\pi$ we have exactly the same scaling of $U_\pi^{(2)}(\tau_a^{(2)})$ and of $\pi_{S^1_\pm}(W_\pm(a))$ by the adjoint action of $U_\pi^{(2)}(\delta_s^{(2)})$. Thus we find that

$$Ad\left(U_\pi^{(2)}(\delta_s^{(2)})\right)(Z_\pi(a)) = Z_\pi(e^s a), \tag{17}$$

but on the other hand of course, as Z_π is in the center, the left hand side should be simply equal to $Z_\pi(a)$. So $Z_\pi(a) = Z_\pi(e^s a)$ for all values of the parameters a and s which means that Z_π is trivial and hence in the projective sense $W_\pi(a)$ equals to $U_\pi^{(2)}(\tau_a)$. $\square$

Corollary 3.6. *The projective representation $U_\pi^{(2)}$ is of positive energy.*

Proof. As the spectrum of the generator of a one-parameter unitary group remains unchanged in any normal representation, by Prop. 3.2 the selfadjoint generator of the one-parameter group

$$a \mapsto \pi_{S^1_+}\left(e^{iaT(t^{(2)}_+)}\right)\pi_{S^1_-}\left(e^{iaT(t^{(2)}_-)}\right) \tag{18}$$

is bounded from below and by Prop. 3.4 this one-parameter group of unitaries equals to the one-parameter group $a \mapsto U^{(2)}_\pi(\tau^{(2)}_a)$ in the projective sense. So by Lemma 3.1 the representation $U^{(2)}_\pi$ is of positive energy. $\square$

Let us now take an arbitrary positive integer n. By equation (6) $R_\alpha \in$ Möb$^{(n)}$ for all $\alpha \in \mathbb{R}$, and by definition both $U^{(n)}_\pi(R_\alpha)$ and $U^{(2)}_\pi(R_\alpha)$ implement the same automorphism of $\pi(\mathcal{A})$. Since both unitaries are actually elements of $\pi(\mathcal{A}_U) \subset \pi(\mathcal{A})$, they must commute and

$$C^{(n)}_\pi(\alpha) = (U^{(n)}_\pi(R_\alpha))^*\, U^{(2)}_\pi(R_\alpha) \tag{19}$$

is a strongly continuous one-parameter group in $\mathcal{Z}(\pi(\mathcal{A}))\cap\pi(\mathcal{A}_U) \subset \mathcal{Z}(\pi(\mathcal{A}_U))$.

As it was mentioned, by [5, Theorem A.1] the restriction of the subnet $\mathcal{A}_U$ onto $\mathcal{H}_{\mathcal{A}_U}$ — unless $\mathcal{A}$ is trivial, in which case $\dim(\mathcal{H}_{\mathcal{A}_U}) = \dim(\mathcal{H}_{\mathcal{A}}) = 1$ — is isomorphic to a Virasoro net. Thus $\mathcal{H}_{\mathcal{A}_U}$ must be separable (even if the full Hilbert space $\mathcal{H}_{\mathcal{A}}$ is not so; recall that we did not assume separability) as the Hilbert space of a Virasoro net is separable.

Every von Neumann algebra on a separable Hilbert space has a strongly dense separable C^* subalgebra. A von Neumann algebra generated by a finite number of von Neumann algebras with strongly dense separable C^* subalgebras has a strongly dense C^* subalgebra. Thus considering that for an $I \in \mathcal{I}$ the restriction map from $\mathcal{A}_U(I)$ to $\mathcal{A}_U(I)|_{\mathcal{H}_{\mathcal{A}_U}}$ is an isomorphism, one can easily verify that the von Neumann algebra $\pi(\mathcal{A}_U)$ has a strongly dense C^* subalgebra.

We can thus safely consider the direct integral decomposition of $\pi(\mathcal{A}_U)$ along its center

$$\pi(\mathcal{A}) = \int_X^{\oplus} \pi(\mathcal{A})(x)d\mu(x). \tag{20}$$

(Even if $\mathcal{H}_\pi$ is not separable, by the mentioned property of the algebra $\pi(\mathcal{A}_U)$, it can be decomposed into the direct sum of invariant separable subspaces for $\pi(\mathcal{A}_U)$. Then writing the direct integral decomposition in each of those subspaces, the rest of the argument can be carried out without further changes.)

For an introduction on the topic of the direct integrals see for example [21, Chapter 14.].

As it was mentioned the representations $U_\pi^{(n)}$ $(n = 1, 2, ..)$ have a unique strongly continuous lift $\tilde{U}_\pi^{(n)}$ to $\widetilde{\text{Möb}}$ where $\tilde{U}_\pi^{(n)}$ is a true representation. Since the group in question is in particular second countable and locally compact, and all these representations are in $\pi(\mathcal{A}_U)$, the decomposition (20) also decomposes these representations (cf. [8, Lemma 8.3.1 and Remark 18.7.6]):

$$\tilde{U}_\pi^{(n)}(\cdot) = \int_X^\oplus \tilde{U}_\pi^{(n)}(\cdot)(x) d\mu(x) \tag{21}$$

where $\tilde{U}_\pi^{(n)}(\widetilde{\text{Möb}})(x) \subset \pi(\mathcal{A})(x)$ and $\tilde{U}_\pi^{(n)}(\cdot)(x)$ is a strongly continuous representation for almost every $x \in X$.

Lemma 3.7. *The representation $\tilde{U}_\pi^{(n)}$ is of positive energy if and only if $\tilde{U}_\pi^{(n)}(\cdot)(x)$ is of positive energy for almost every $x \in X$.*

Proof. For a $t \mapsto V(t)$ strongly continuous one-parameter group of unitaries the positivity of the selfadjoint generator is for example equivalent with the fact that $\hat{V}(f) \equiv \int V(t) f(t) dt = 0$ for a certain smooth, fast decreasing function f whose Fourier transform is positive on $\mathbb{R}^-$ and zero on $\mathbb{R}^+$. If V is a direct integral of a measurable family of strongly continuous one-parameter groups, $V(\cdot) = \int_X^\oplus V(\cdot)(x) d\mu(x)$, then $\hat{V}(f) = \int_X^\oplus \hat{V}(f)(x) d\mu(x)$. As $\hat{V}(f)(x) \geq 0$ for almost every $x \in X$, the operator $\hat{V}(f)$ is zero if and only if $\hat{V}(f)(x) = 0$ for almost every $x \in X$. $\square$

As $C_\pi^{(n)}$ is a strongly continuous one-parameter group in the center, for almost all $x \in X$: $\tilde{U}_\pi^{(n)}(R_{(\cdot)})(x) = \tilde{U}_\pi^{(2)}(R_{(\cdot)})(x)$ in the projective sense. Therefore, since by Lemma 3.7 and Corollary 3.6 in $\tilde{U}_\pi^{(2)}(\cdot)(x)$ the selfadjoint generator of rotations is positive, also in $\tilde{U}_\pi^{(n)}(\cdot)(x)$ it must be at least bounded from below and hence by Lemma 3.3 it is actually positive. Thus, by using again Lemma 3.7 we arrive to the following result.

Theorem 3.8. *Let π be a locally normal representation of the conformal local net of von Neumann algebras on the circle $(\mathcal{A}, U)$. Then the strongly continuous projective representation $U_\pi^{(n)}$ of $\text{Möb}^{(n)}$, defined by Proposition 3.3, is of positive energy for all positive integers n. In particular, the unique continuous inner implementation of the Möbius symmetry in the representation π is of positive energy.*

Carpi proved [5, Prop. 2.1] that an irreducible representation of a Virasoro net $\mathcal{A}_{\mathrm{Vir},c}$ must be one of those that we get by integrating a positive energy unitary representation of the Virasoro algebra (corresponding to the same central charge) under the condition that the representation is of positive energy. Thus by the above theorem we may draw the following conclusion.

Corollary 3.9. *An irreducible representation of the local net $\mathcal{A}_{\mathrm{Vir},c}$ must be one of those that we get by integrating a positive energy unitary representation of the Virasoro algebra corresponding to the same central charge.*

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